

Rainbow Cycles and Paths

Bachelor Thesis
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Abstract

In a properly edge colored graph, a subgraph using every color at most once is called *rainbow*. In this thesis, we study rainbow cycles and paths in proper edge colorings of complete graphs, and we prove that in every proper edge coloring of K_n , there is a rainbow path on $(3/4 - o(1))n$ vertices, improving on the previously best bound of $(2n + 1)/3$ from [?]. Similarly, a k -rainbow path in a proper edge coloring of K_n is a path using no color more than k times. We prove that in every proper edge coloring of K_n , there is a k -rainbow path on $(1 - 2/(k + 1)!)n$ vertices.

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1 Introduction

1.1 Rainbow cycles and paths

Consider an edge colored graph G . A subgraph of G is called *rainbow* (or *heterochromatic*) if no two of its edges receive the same color. We are concerned with rainbow paths and, to a lesser extent, cycles in proper edge colorings of the complete graph K_n . Hahn conjectured that every proper edge coloring of K_n admits a Hamiltonian rainbow path (a rainbow path visiting every vertex of K_n) (c.f. [?]). Maamoun and Meyniel [?] disproved this conjecture by constructing counterexamples for the case where n is a power of two, as follows. Let $n = 2^m$. Then we can identify the vertices of K_n with distinct elements of the group $(\mathbb{Z}/2\mathbb{Z})^m$, and color every edge $\{a, b\}$ of K_n with the sum of the group elements corresponding to a and b . This is a proper edge coloring, because for two edges $\{a, b\}$ and $\{a, c\}$, the group property implies $a + b \neq a + c$. Maamoun and Meyniel proved that this coloring admits no Hamiltonian rainbow paths. The reader is invited to check this fact for the case of K_4 .

Conversely, it is widely believed that in every proper edge coloring of K_n , there is a rainbow path on $n - 1$ vertices (see for example [?]). Still, this is far from proved, and to date, the best general lower bound on the number of vertices in a maximum rainbow path in (a properly edge colored) K_n is $(2n + 1)/3$, as proved by Gyárfás and Mhalla in [?]. The main result of this thesis improves this bound to $(3/4 - o(1))n$.

Theorem 1. In every proper edge coloring of K_n , there is a rainbow path of length

$$\left(\frac{3}{4} - o(1)\right)n.$$

Several theorems and conjectures on rainbow cycles can be found in a paper by Akbari, Etesami, Mahini and Mahmoody [?]. Most importantly (for our purposes), it is proved that in every proper edge coloring of K_n , there is a rainbow cycle of length at least $n/2 - 1$. This result was later improved on by Gyárfás, Ruszinkó, Sarközy and Schelp in [?], where a bound of $(4/7 - o(1))n$ is given.

A related topic is that of colorful Hamiltonian cycles in proper edge colorings of K_n . In [?], it is conjectured that every proper edge coloring of K_n contains a Hamiltonian cycle using at least $n - 2$ colors, and it is proved that there is always one with at least $(2/3 - o(1))n$ colors. The construction used in our proof of Theorem 1 can be used to show that there are Hamiltonian cycles using at least $(3/4 - o(1))n$ colors.

In [?], Hahn and Thomassen studied rainbow cycles and paths in *k*-bounded edge colorings of K_n , that is, (not necessarily proper) edge colorings that use every color at most *k* times. It is shown that for fixed *k* and large enough *n*, every such coloring contains a Hamiltonian rainbow path; and the authors conjecture that there are Hamiltonian rainbow paths even if $k = an$ for some suitably small constant factor *a*.

1.2 Paths with repeated colors

Generalizing the notion of a rainbow path, we consider paths in K_n that use every color at most a constant number of times. We call such paths *k*-rainbow paths, where *k* is the number of times a color may appear on the path. We will prove that in every proper edge coloring of K_n and for every integer $k > 0$, there are *k*-rainbow paths on at least $n - O(n/k!)$ vertices. As far as we know, there are no previous results in this direction.

1.3 Note on Latin squares

We now give some motivation for the study of rainbow cycles and paths by relating it to problems whose nature is not inherently graph-theoretic.

Latin squares have been a popular topic in combinatorics at least since the times of Euler, who studied them extensively. An array of *n* rows and *n* columns is called a *Latin square of order n* if every number in $[n] = \{1, \dots, n\}$ appears exactly once in each of its rows and columns. A (complete) *transversal* of a Latin square is a selection of *n* cells of the square, choosing exactly one from each row and column, such that every number in $[n]$ is contained in exactly one cell of the selection. Similarly, a *partial transversal* of a Latin square is a maximal selection of cells, each cell again being from a different row and column, such that no two chosen cells contain the same symbol.

As shown by Maillet (1894), there are many Latin squares which do not have complete transversals. However, a famous conjecture of Ryser (1967) states that every Latin square of odd order has a transversal and, moreover, Brualdi conjectured that every Latin square of order *n* admits a partial transversal of size at least $n - 1$.

Proofs for both conjectures seem out of reach, even though it is known that in every Latin square of order n , there are partial transversals of size $n - o(n)$. All this and more can be found in [?].

How do Latin squares relate to rainbow cycles, or graph theory in general? Consider any Latin square A , and let R and C denote the sets of its rows and columns respectively. Then A defines a proper edge coloring of the complete bipartite graph with partite sets C and R , as follows: for $c \in C$ and $r \in R$, the edge $\{c, r\}$ is colored with the number contained in the cell determined by the row r and the column c . Then a transversal of A corresponds to a perfect bipartite matching in this graph, in which no two edges use the same color: a rainbow perfect matching. Rainbow matchings are studied for example in [?]. Conversely, every proper edge coloring of K_n with $n - 1$ colors can be used to construct a Latin square A of order n , in the following way. Let $\{v_1, \dots, v_n\}$ denote the vertex set of K_n . Then for every $i \in [n]$ and $j \neq i$, let $A_{i,i} = n$ and let $A_{i,j}$ be the color of the edge $\{v_i, v_j\}$. A complete transversal of this Latin square corresponds to a 2-regular rainbow subgraph (i.e., a subgraph consisting of vertex-disjoint cycles) that covers all but at most one vertex of K_n . This can be seen as follows. For every vertex v_i , either $A_{i,i}$ is in the transversal, or two cells $A_{i,j}$ and $A_{k,i}$ (with $i \neq j$ and $i \neq k$) are in the transversal. In the former case, v_i does not belong to the subgraph. In the latter case, the two edges $\{v_i, v_j\}$ and $\{v_i, v_k\}$ are included in the subgraph. Hence, all included vertices have degree two. By the defining property of a transversal, all selected edges have different colors, so the subgraph is really rainbow, and there are no cycles of length two. Moreover, for every vertex v_i , we have $A_{i,i} = n$, so at most one vertex does not belong to the subgraph.

1.4 Thesis structure

In Chapter 2, some results on rainbow paths are proved; in particular, we prove Theorem 1. The chapter also contains an informal overview over the ideas used in the proof.

Chapter 3 takes a look at k -rainbow paths. It is proved that every proper edge coloring of K_n contains a k -rainbow path on at least $(1 - 2/(k+2)!)n$ vertices. The final chapter is a short conclusion.

1.5 Notation

In this section, we define the notation used throughout the thesis and briefly introduce the basic graph-theoretic notions.

Sets

We write N for the set $\{1, 2, 3, \dots\}$ of natural numbers. If n is a natural number, then we write $[n]$ for the set $\{1, 2, \dots, n\}$.

We use capital letters for sets. If A is a set, then $|A|$ is the cardinality of A and $\binom{A}{k}$ is the set of all k -element subsets of A . We write A^c for the complement of A (relative to some universe).

Graphs

For graphs, we follow the notation from [?], although we will restate the most important definitions here.

A *graph* is a tuple (V, E) , where V is the finite set of *vertices* and $E \subseteq \binom{V}{2}$ is the set of *edges*. The *endpoints* of an edge are its elements, and the edge is *incident* to them and only them. Two edges are *coincident* if they intersect, and two vertices u and v are *adjacent* if $\{u, v\} \in E$. If G is a graph, then we write $V(G)$ for its vertex set and $E(G)$ for its edge set. A graph $H = (V', E')$ is a *subgraph* of $G = (V, E)$, written $H \subseteq G$, if $V' \subseteq V$ and $E' \subseteq E$.

For a graph $G = (V, E)$ and an arbitrary $e \in \binom{V}{2}$, we write $G + e$ and $G - e$ for the graphs $(V, E \cup \{e\})$ and $(V, E \setminus \{e\})$. If $G = (V, E)$ and $H = (V', E')$ are graphs, then $G \cup H$ denotes the graph $(V \cup V', E \cup E')$.

The *complete graph on n vertices* is the graph with vertex set $[n]$ and edge set $\binom{[n]}{2}$. It is denoted by K_n .

Given a graph $G = (V, E)$, a map $c: E \rightarrow N$ is called a *proper edge coloring* (or simply a *coloring*) of G if for every two coincident edges e and e' of G , we have $c(e) \neq c(e')$. The colors in the image domain $c(E)$ of c are called the colors *used* by G , and we usually write $c(G)$ for this set. For an edge $\{u, v\} \in E$, we usually write $c(u, v)$ instead $c(\{u, v\})$. Slightly abusing this notation, if A is a set of vertices, then we also write $c(u, A)$ for the set $c(E(u, A))$.

Cycles and paths

A *path* is a non-empty graph $P = (V, E)$ of the form

$$V = \{p_1, p_2, \dots, p_k\} \quad \text{and} \quad E = \{\{p_1, p_2\}, \{p_2, p_3\}, \dots, \{p_{k-1}, p_k\}\},$$

which we usually denote by the sequence (p_1, p_2, \dots, p_k) . Then p_1 and p_k are the *start* and *end* vertices of P , respectively. The number of edges in E is called the *length* of P . We call p_i a *k-successor* of p_j if $i > j$ and there are at most $k - 1$ vertices between p_i and p_j on P . In other words, p_i is a *k-successor* of p_j if $0 < i - j \leq k$. Equivalently, p_j is a *k-predecessor* of p_i .

If $P = (p_1, p_2, \dots, p_k)$ is a path, then the graph $C = P + \{p_k, p_1\}$ is a *cycle*, and $|E(C)|$ is the *length* of C . We represent this cycle by the cyclic sequence of its

vertices, for example $C = (p_1, p_2, \dots, p_k, p_1)$. If G is a graph and $H \subseteq G$ is a path or cycle such that $V(H) = V(G)$, then H is called *Hamiltonian*.

2 Rainbow Paths

2.1 Introduction

Consider the complete graph $K_n = (V, E)$ with a proper edge coloring $c: E \rightarrow N$. Given this coloring, the rainbow paths and cycles are exactly the paths and cycles in K_n that use every color at most once.

If P is a rainbow path, then we will refer to the colors in $c(P)$ as *old* and to those in $c(P)^c = c(E) \setminus c(P)$ as *new*. Edges colored with old colors are *old*, edges colored with new colors are *new*.

In [?], Gyárfás and Mhalla proved that regardless of how the coloring is chosen, there always are rainbow paths on at least $(2n + 1)/3$ vertices. Now we give the basic idea behind their proof, the details of which we will see later. Consider any maximum rainbow path $P = (p_1, \dots, p_k)$ in K_n , and consider an edge $\{p_i, p_{i+1}\}$ such that $\{p_1, p_{i+1}\}$ is new, as in the following figure.

Clearly, any edge $\{p_k, r\}$ with $r \in V(P)^c$ cannot use the color of $\{p_{i+1}, p_i\}$, as otherwise the path

$$(p_i, \dots, p_1, p_{i+1}, \dots, p_k, r)$$

would be a rainbow path on $|V(P)| + 1$ vertices, contradicting the choice of P . Viewed the other way around, we can say that a certain number of edges in $E(P)$ are not allowed to have colors in $c(p_k, V(P)^c)$. But all the edges in $E(p_k, V(P)^c)$ must be old, so their colors appear somewhere on the path. As Gyárfás and Mhalla observed, this conflict leads to the bound $k \geq (2n + 1)/3$. But what if we knew that starting in any vertex $r \in V(P)^c$, there is a rainbow path in $V(P)^c$ (of a certain minimum length l) that uses no colors of $c(P)$? Assume that this is the case, and, moreover, that given two arbitrary new colors, this path can be chosen in such a way that it does not use any one of them. Then instead of forbidding colors in $c(p_k, V(P)^c)$ to appear only on edges $\{p_i, p_{i+1}\}$ as above, we can also forbid them to appear on any edge $\{p_i, p_{i+1}\}$ such that p_{i+1} has an l -successor p_j with $c(p_1, p_j) \notin c(P)$. While this does not immediately lead to a good bound on the length of P , it does give us more flexibility; as we will see we can now often simply ‘forget’ about constant terms. This is the main idea behind the upcoming proof that in every proper edge coloring of K_n , there are rainbow paths of length $(3/4 - o(1))n$.

Figure 2.1: Vertices in $A(P)$ (circled) and $B(P)$ (framed)

Now for some definitions. For any vertex $v \in V$, we define the *new neighborhood* of v relative to a rainbow path P by

$$\Gamma_{\text{new}}(v, P) = \{u \in V \setminus \{v\} : c(u, v) \notin c(P)\}.$$

Analogously, if C is a rainbow cycle, then

$$\Gamma_{\text{new}}(v, C) = \{u \in V \setminus \{v\} : c(u, v) \notin c(C)\}.$$

Moreover, for any rainbow path $P = (p_1, \dots, p_k)$, we define the sets

$$A(P) = \{p_i \in V(P) : p_{i+1} \in \Gamma_{\text{new}}(p_1, P)\}$$

and

$$B(P) = \{p_i \in V(P) : p_{i-1} \in \Gamma_{\text{new}}(p_k, P)\}.$$

Note that these definitions are symmetric in the sense that if $P' = (p_k, \dots, p_1)$, then $A(P) = B(P')$ and $A(P') = B(P)$. Figure 2.1 serves as a visual aid for the formal definitions of $A(P)$ and $B(P)$.

We will be working mostly with maximum rainbow paths, that is, rainbow paths of maximum length. Clearly, if $P = (p_1, \dots, p_k)$ is a maximum rainbow path, then adding any edge from $E(p_k, V(P)^c)$ to it cannot result in a rainbow path (otherwise P would not be maximum). This means that all edges in $E(p_k, V(P)^c)$ use colors that are also used by P . The same argument can be made for edges in $E(p_1, V(P)^c)$.

Proposition 1. If $P = (p_1, \dots, p_k)$ is a maximum rainbow path with respect to some coloring c , then $c(p_1, V(P)^c) \subseteq c(P)$ and $c(p_k, V(P)^c) \subseteq c(P)$. In particular, $|A(P)| = |\Gamma_{\text{new}}(p_1, P)| \geq n - k$ and $|B(P)| = |\Gamma_{\text{new}}(p_k, P)| \geq n - k$.

Using this simple observation, we directly obtain the following lower bound on the length of a maximum rainbow path.

Proposition 2. In every proper edge coloring of K_n , there are rainbow paths on at least $(n + 1)/2$ vertices.

Proof. Consider an arbitrary proper edge coloring c of K_n , and let $P = (p_1, \dots, p_k)$ be a maximum rainbow path in this coloring. We have

$$|c(p_k, V(P)^c)| = |V(P)^c| = n - k,$$

and

$$|c(P)| = |E(P)| = k - 1.$$

Figure 2.2: The path $\rho_i \cdot (p_1, \dots, p_k)$

By Proposition 1, we have $c(p_k, V(P)^c) \subseteq c(P)$, and hence $|c(p_k, V(P)^c)| \leq |c(P)|$. Thus $n - k \leq k - 1$, and so we have

$$k \geq \frac{n+1}{2},$$

as claimed. \square

Actually, we proved something slightly stronger, namely, that given any vertex v of K_n , there is a rainbow path on $(n+1)/2$ vertices starting in this vertex. To see this, revisit the proof and let P be the longest rainbow path starting in v , and observe that the argument made for Proposition 1 still applies.

2.2 Rotations

The paths in K_n admit many symmetries; in fact, every permutation of the vertices of a path results in another path. We consider a special kind of permutation, which we call rotation here. Rotations were already used by Pósa in [?]. For every $i \in [k]$, there is a rotation ρ_i which acts on the path $P = (p_1, \dots, p_k)$ to produce the path

$$\rho_i \cdot P = (p_i, p_{i-1}, \dots, p_1, p_{i+1}, p_{i+2}, \dots, p_k),$$

as shown in figure 2.2.

The point is that if $P = (p_1, \dots, p_k)$ is a rainbow path and $p_i \in A(P)$, then $\rho_i \cdot P$ is a rainbow path that does not use the color $c(p_i, p_{i+1})$, but that is still very similar to P . In particular, $\rho_i \cdot P$ ends in the same vertex as P .

Proposition 3. If $P = (p_1, \dots, p_k)$ is a maximum rainbow path with respect to some coloring c , then for every $p_i \in A(P)$ we have $c(p_i, p_{i+1}) \notin c(p_k, V(P)^c)$. Similarly, for every $p_i \in B(P)$ we have $c(p_i, p_{i-1}) \notin c(p_1, V(P)^c)$.

Proof. Suppose that $p_i \in A(P)$ is such that $c(p_i, p_{i+1}) \in c(p_k, V(P)^c)$. Then there is a vertex $v \in V(P)^c$ such that $c(p_k, v) = c(p_i, p_{i+1}) \notin c(\rho_i \cdot P)$. Since $\rho_i \cdot P$ ends in p_k and $v \notin V(P) = V(\rho_i \cdot P)$, the rainbow path $\rho_i \cdot P$ violates Proposition 1.

The second part follows by symmetry. \square

This fact was used by Gyárfás and Mhalla to find rainbow paths on $(2n+1)/3$ vertices. Because the proof is quite elegant and uses a technique similar to those used later on, we give it here.

Theorem 2 [?]. In every proper edge coloring of K_n , there is a rainbow path on at least $(2n + 1)/3$ vertices.

Proof. Consider an arbitrary proper edge coloring c of K_n , and let $P = (p_1, \dots, p_k)$ be a maximum rainbow path with respect to c . By Proposition 1, $c(p_k, V(P)^c) \subseteq c(P)$. Now let

$$X = \{c(p_i, p_{i+1}) : p_i \in A(P)\}.$$

Then we have $|X| = |A(P)| \geq n - k$. Since $X \subseteq c(P)$, we get

$$X \cup c(p_k, V(P)^c) \subseteq c(P),$$

and hence $|X \cup c(p_k, V(P)^c)| \leq |c(P)| = |E(P)| = k - 1$. By Proposition 3, we have $|X \cap c(p_k, V(P)^c)| = \emptyset$, and thus

$$|X \cup c(p_k, V(P)^c)| = |X| + |c(p_k, V(P)^c)| \geq n - k + n - k,$$

and so $2n - 2k \leq k - 1$, or

$$k \geq \frac{2n + 1}{3},$$

which is what we needed to prove. □

Now we come to the main result of this work.

2.3 A theorem on the length of rainbow paths

In this section, we are going to prove the following result, which is equivalent to Theorem 1.

Theorem 3. For every $\epsilon > 0$, there is some $n_0 = n_0(\epsilon)$ such that for $n > n_0$, every proper edge coloring of K_n contains a rainbow path on at least $(3/4 - \epsilon)n$ vertices.

The proof proceeds by contradiction, that is, we assume that for some $\epsilon > 0$, there are no such rainbow paths, and show that we can construct a rainbow path that is longer than the length of a (supposedly) maximum rainbow path. There are two steps to how this is done:

1. We show that there is a maximum rainbow path P such that almost all (i.e., all but constantly many) vertices in $V(P)^c$ have at least some constant number of new neighbors in $V(P)^c$.
2. We show how such a rainbow path can be extended to a longer rainbow path, resulting in a contradiction.

Preliminaries

Let $\epsilon > 0$ and let $n > n_0$ for some suitably large value $n_0 = n_0(\epsilon)$. We will not give an explicit value for n_0 , rather we will tacitly assume that increasing functions of n dominate any constant. We are given a proper edge coloring c of K_n . Then we denote by t the number of vertices in a maximum length rainbow path, that is, there are no rainbow paths of length t (on $t + 1$ vertices). We will assume throughout that we have

$$t \leq \left(\frac{3}{4} - \epsilon \right) n,$$

trying to arrive at a contradiction.

During the proof, let a be a ‘large enough’ constant. For example $a = 100 + \lceil 100/\epsilon \rceil$ is more than enough.

Nice rainbow paths

For any rainbow path P , we define the set

$$R(P) = \{r \in V(P)^c : |\Gamma_{\text{new}}(r, P) \cap V(P)^c| > a\}.$$

So $R(P)$ is the set of vertices in $V(P)^c$ that have more than a new neighbors in $V(P)^c$. Then we have the following definition.

Definition 1 (Nice rainbow path). A *nice* rainbow path is a rainbow path P satisfying

$$|R(P)| > n - t - 1/\epsilon.$$

In other words, nice rainbow paths are such that all but at most $1/\epsilon$ vertices in $V(P)^c$ have more than a new neighbors in $V(P)^c$. We will be interested in nice maximum rainbow paths, that is, nice rainbow paths on t vertices. Before showing that such paths exist, we will motivate them by proving that they have some nice properties.

Proposition 4. If P is nice maximum rainbow path, then for any vertex $r \in R(P)$ and any set F of colors, there is a rainbow path Q starting in r in the subgraph induced by $R(P)$ such that $|V(Q)| = \lfloor (a - |F| - 1/\epsilon)/2 \rfloor$ and $c(Q) \cap (F \cup c(P)) = \emptyset$.

Proof. Let $k = \lfloor (a - |F| - 1/\epsilon)/2 \rfloor$. We construct a series of rainbow paths Q_1, \dots, Q_k starting in r and using only vertices in $R(P)$, such that for every $i \in [k]$, we have $|V(Q_i)| = i$. Furthermore, we shall make sure that no Q_i uses colors in $F \cup c(P)$. Clearly, the path Q_k will have the desired properties.

Let $Q_1 = (r)$ and define Q_{i+1} in terms of Q_i as follows. By construction, Q_i starts in r and ends in some vertex $v \in R(P)$. If there is a vertex $u \in \Gamma_{\text{new}}(v, P) \cap R(P) \setminus V(Q_i)$ such that

$$c(u, v) \notin c(Q_i) \cup F,$$

then we can simply define Q_{i+1} to be $Q_i \cup (v, u)$. Hence it is enough to prove that $|\Gamma_{\text{new}}(v, P) \cap R(P) \setminus V(Q_i)| > |c(Q_i) \cup F|$.

Because $i < k$, we have

$$|c(Q_i) \cup F| \leq |c(Q_i)| + |F| < k + |F|.$$

Since P is nice, we have $|V(P)^c \setminus R(P)| < 1/\epsilon$, and as $v \in R(P)$, we get

$$|\Gamma_{\text{new}}(v, P) \cap R(P)| > a - 1/\epsilon,$$

and hence

$$|\Gamma_{\text{new}}(v, P) \cap R(P) \setminus V(Q_i)| > a - 1/\epsilon - k.$$

Now the claim follows from

$$a - 1/\epsilon - k \geq k + |F|,$$

which holds because $k \leq (a - |F| - 1/\epsilon)/2$. □

The importance of this fact comes from the following lemma.

Lemma 1. If P is a nice maximum rainbow path, then every rainbow path

$$Q = (q_1, \dots, q_k)$$

satisfies at least one of the following properties:

- (P1) $V(Q) \not\subseteq V(P)$,
- (P2) $(\Gamma_{\text{new}}(q_1, Q) \cup \Gamma_{\text{new}}(q_k, Q)) \cap R(P) = \emptyset$,
- (P3) $|c(Q) \setminus c(P)| > 2$, or
- (P4) $k < t - a/3$.

Proof. Suppose that there is a rainbow path $Q = (q_1, \dots, q_k)$ satisfying none of the properties (P1–4).

Because (P2) is violated, either $\Gamma_{\text{new}}(q_1, Q) \cap R(P) \neq \emptyset$ or $\Gamma_{\text{new}}(q_k, Q) \cap R(P) \neq \emptyset$. In any case, there is an endpoint q of Q such that for some vertex $s_1 \in R(P)$, we have $c(q, s_1) \notin c(Q)$. Then let

$$F = \{c(q, s_1)\} \cup (c(Q) \setminus c(P)),$$

and since (P3) is violated, we have $|F| \leq 3$.

By Proposition 4, we know that the subgraph induced by $R(P)$ contains a rainbow path $S = (s_1, \dots, s_l)$ with $l \geq \lfloor (a - 3 - 1/\epsilon)/2 \rfloor$ such that $c(S) \cap (F \cup c(P)) = \emptyset$. So S does not use any colors in

$$F \cup c(P) = \{c(q, s_1)\} \cup c(Q) \cup c(P),$$

and hence the path $S' = (q_1, s_1, \dots, s_l)$ is a rainbow path with $c(S') \cap c(Q) = \emptyset$.

Because (P1) is violated, we have $V(Q) \cap R(P) = \emptyset$, and thus

$$V(Q) \cap V(S') = \{q\}.$$

so $Q \cup S'$ is a rainbow path. As (P4) is violated, we have

$$\begin{aligned} |V(Q \cup S')| &= |V(Q)| + |V(S')| - 1 \\ &= k + l - 1 \\ &\geq t - \frac{a}{3} + \lfloor (a - 3 - 1/\epsilon)/2 \rfloor - 1 \\ &> t - \frac{a}{3} + \frac{a}{2} - \frac{1}{2\epsilon} - 3 \\ &> t. \end{aligned}$$

But by definition of t , there are no rainbow paths this long. \square

Thus Lemma 1 suggests that we can prove Theorem 3 in two steps. Under the assumption that $t \leq (3/4 - \epsilon)n$:

- Show that there is at least one nice maximum rainbow path P .
- Then show that for this P , there is a rainbow path Q violating Lemma 1.

Existence of nice maximum rainbow paths

In this section, we will prove that there are nice maximum rainbow paths. First, let us reiterate what it means for a rainbow path not to be nice. If P is a rainbow path that is not nice, then we have

$$|R(P)| \leq n - t - 1/\epsilon,$$

or, equivalently,

$$|R(P)^c| \geq n - (n - t - 1/\epsilon) = t + 1/\epsilon.$$

For any rainbow path, we have $|V(P)| \leq t$, so we get

$$|R(P)^c \cap V(P)^c| = |R(P)^c \setminus V(P)| \geq 1/\epsilon.$$

This means that there are at least $1/\epsilon$ vertices in $V(P)^c$ that have no more than a new neighbors in $V(P)^c$.

Figure 2.3: A rainbow cycle if $p_i \in A(P) \cap B(P)$ and $c(p_1, p_{i+1}) \neq c(p_{i-1}, p_t)$

Proposition 5. Suppose that there are no nice maximum rainbow paths. Then there are no rainbow cycles of length t .

Proof. Suppose that C is such a cycle. Then removing any edge $e \in E(C)$ from C , we get a maximum rainbow path $P = C - e$, with $V(P) = V(C)$.

By assumption, this maximum rainbow path cannot be not nice, so there is a vertex $v \in V(C)^c$ such that at least $|\Gamma_{\text{new}}(v, P)| - a \geq n - t - a$ new neighbors of v are in $V(C)$. Since, for large enough n , we have $n - t - a \geq 2$, at least one of those neighbors, call it u , is such that $c(u, v) \neq c(e)$. So $c(u, v) \notin c(P + e) = c(C)$.

Let u' be a vertex adjacent to u on C . Then $(C - \{u, u'\}) \cup (u, v)$ is a rainbow path of length t . This contradicts the definition of t . \square

Proposition 6. Suppose that there are no nice maximum rainbow paths, and let $P = (p_1, \dots, p_t)$ be a maximum rainbow path. Then for every vertex $p_i \in A(P) \cap B(P)$ we have $c(p_1, p_{i+1}) = c(p_{i-1}, p_t)$.

Proof. Assume by way of contradiction that there is a vertex $p_i \in A(P) \cap B(P)$ such that $c(p_1, p_{i+1}) \neq c(p_{i-1}, p_t)$. By definition, $c(p_1, p_{i+1}) \notin c(P)$ and $c(p_t, p_{i-1}) \notin c(P)$, so the cycle

$$C = (p_1, \dots, p_{i-1}, p_t, p_{t-1}, \dots, p_{i+1}, p_1)$$

shown in figure 2.3 is a rainbow cycle of length $t - 1$.

By assumption, P could not have been nice, so there is a vertex $v \in V(P)^c$ with

$$|\Gamma_{\text{new}}(v, P) \cap V(P)| \geq |\Gamma_{\text{new}}(v, P)| - a.$$

Some thought shows that

$$|\Gamma_{\text{new}}(v, C) \cap V(C)| \geq |\Gamma_{\text{new}}(v, C)| - a - 5,$$

because at most three vertices of $\Gamma_{\text{new}}(v, P) \cap V(P)$ are in not in $\Gamma_{\text{new}}(v, C) \cap V(C)$, and at most two vertices of $\Gamma_{\text{new}}(v, C)$ are not in $\Gamma_{\text{new}}(v, P)$.

But no two vertices $u, u' \in \Gamma_{\text{new}}(v, C) \cap V(C)$ can be adjacent in C , because otherwise, the cycle $(C - \{u, u'\}) \cup (u, v, u')$ would be a rainbow cycle of length t , contradicting Proposition 5. In other words, no edge $e \in E(C)$ is incident to two vertices in $\Gamma_{\text{new}}(v, C) \cap V(C)$. Hence, if we write $X = \{e \in E(C) : e \cap \Gamma_{\text{new}}(v, C) \neq \emptyset\}$, then

$$\begin{aligned} |X| &= 2|\Gamma_{\text{new}}(v, C) \cap V(C)| \\ &\geq 2|\Gamma_{\text{new}}(v, C)| - 2a - 10 \\ &\geq 2n - 2t - 2a - 12. \end{aligned}$$

Now let $Y = V(C)^c \cap \Gamma_{\text{new}}(v, C)^c$. We have

$$\begin{aligned}
 |Y| &= |V(C)^c \cap \Gamma_{\text{new}}(v, C)^c| \\
 &= |V(C)^c| - |\Gamma_{\text{new}}(v, C) \cap V(C)^c| \\
 &= n - t + 1 - |\Gamma_{\text{new}}(v, C)| + |\Gamma_{\text{new}}(v, C) \cap V(C)| \\
 &\geq n - t + 1 - |\Gamma_{\text{new}}(v, C)| + |\Gamma_{\text{new}}(v, C)| - a - 5 \\
 &= n - t - a - 4.
 \end{aligned}$$

Clearly, $c(X) \subseteq c(C)$ and $c(v, Y \setminus \{v\}) \subseteq c(C)$. Then $c(X)$ and $c(v, Y \setminus \{v\})$ have to intersect, because

$$\begin{aligned}
 |c(X)| + |c(v, Y \setminus \{v\})| &= |X| + |Y| - 1 \\
 &\geq 3n - 3t - 3a - 17 \\
 &> t - 1 \\
 &= |c(C)|.
 \end{aligned}$$

Therefore, there exists an edge $e \in X$ with $c(e) = c(v, u)$ for some $u \in V(C)^c$. By definition, e is incident to some vertex $x \in V(C)$ with $c(x, v) \notin c(C)$. Then the rainbow path $(C - e) \cup (x, v, u)$ is a rainbow path of length t , but this contradicts the definition of t . \square

This suggests that one could choose P in such a way that $|A(P) \cap B(P)|$ is small. Indeed, we can do this, as we will show next.

Proposition 7. Suppose that there are no nice maximum rainbow paths. Then there is a maximum rainbow path P such that $|A(P) \cap B(P)| \leq \epsilon n$.

Proof. Let $k = \lceil 1 + 2/\epsilon \rceil$ and let P be an arbitrary maximum rainbow path. Now we define a sequence of maximum rainbow paths (P_0, \dots, P_k) as follows:

- $P_0 = P$.
- P_l is defined in terms of $P_{l-1} = (p_1, \dots, p_t)$. Because P_{l-1} is a maximum rainbow path, we have $|A(P_{l-1})| \geq n - t > k$. Then choose a vertex $p_i \in A(P_{l-1})$ that is *not* the starting vertex of any of the paths P_0, \dots, P_{l-1} , and let $P_l = \rho_i \cdot P_{l-1}$. Then, because $p_i \in A(P_{l-1})$, the path P_l is a maximum rainbow path starting in p_i and ending in p_t .

By this construction, all paths P_i end in the same vertex p_t , but no two different paths start in the same vertex. We now show that at least one of the paths P_l must satisfy $|A(P_l) \cap B(P_l)| \leq \epsilon n$. By way of contradiction, assume that for every $l \in \{0, \dots, k\}$, we have

$$|A(P_l) \cap B(P_l)| > \epsilon n.$$

For every path $P_l = (p_1, \dots, p_t)$, we define two sets X_l and Y_l . Let X_l be the smallest set that, for every vertex $p_i \in \Gamma_{\text{new}}(p_t, P_l)$, contains the triples (p_i, p_{i+1}, p_{i+2}) and (p_i, p_{i-1}, p_{i-2}) , if those vertices are defined. Observe that $|X_l \setminus X_{l-1}| \leq 4$, because the path P_l is built from P_{l-1} by a single rotation. Then, since $|X_0| \leq 2t$, we get

$$|X_0 \cup \dots \cup X_k| \leq 2t + 4k.$$

Now let Y_l be the smallest set that, for every vertex $p_i \in A(P_l) \cap B(P_l)$, contains the triple (p_{i-1}, p_i, p_{i+1}) . If $p_i \in B(P_l)$, then $p_{i-1} \in \Gamma_{\text{new}}(p_t, P_l)$, so we have $Y_l \subseteq X_l$.

Assuming that n is large enough, we get

$$\frac{|X_0 \cup \dots \cup X_k|}{|Y_l|} < \frac{2t + 4k}{\epsilon n} < 1 + 2/\epsilon \leq k.$$

Thus $k|Y_l| > |X_0 \cup \dots \cup X_k|$. Since $Y_l \subseteq |X_0 \cup \dots \cup X_k|$, and because there are k sets Y_l altogether, this means that there are two sets Y_i and Y_j that intersect, say in the triple (u, v, w) . But if we write $P_i = (p_1, \dots, p_t)$ and $P_j = (p'_1, \dots, p'_{t-1}, p_t)$, then either $c(p_1, w) \neq c(u, p_t)$ or $c(p'_1, w) \neq c(u, p_t)$, contradicting the result above stating that for any maximum rainbow path $P = (p_1, \dots, p_t)$, every vertex $p_i \in A(P) \cap B(P)$ satisfies $c(p_1, p_{i+1}) = c(p_{i-1}, p_t)$. \square

Now we are ready to prove the following.

Lemma 2. There exists at least one nice maximum rainbow path.

Proof. We may assume that there are no nice maximum rainbow paths. Then, by Proposition 7, there is a maximum rainbow path P with $|A(P) \cap B(P)| \leq \epsilon n$. Writing A and B for $A(P)$ and $B(P)$ respectively, this means that

$$|A \cup B| = |A| + |B| - |A \cap B| \geq 2n - 2t - \epsilon n.$$

Observe that every vertex in $A \cup B$ can have at most one new neighbor in $V(P)^c$, as otherwise we would get a rainbow path of length t .

Now suppose that r is a vertex in $V(P)^c$ such that

$$|\Gamma_{\text{new}}(r, P) \cap V(P)| \geq \Gamma_{\text{new}}(r, P) - a \geq n - t - a.$$

For brevity, let us write $X = \Gamma_{\text{new}}(r, P) \cap V(P)$. Then, using $|A \cup B \cup X| \leq t$,

$$\begin{aligned} |(A \cup B) \cap X| &= |A \cup B| + |X| - |A \cup B \cup X| \\ &\geq 2n - 2t - \epsilon n + n - t - a - t \\ &= 3n - 4t - a - \epsilon n \\ &\geq \epsilon n. \end{aligned}$$

Therefore, at least ϵn new neighbors of r are in $A \cup B$. Hence, by the observation above, there may be at most

$$\frac{|A \cup B|}{\epsilon n} \leq \frac{2n - 2t - \epsilon n}{\epsilon n} < \frac{1}{\epsilon}$$

vertices $r \in V(P)^c$ with $|\Gamma_{\text{new}}(r, P) \cap V(P)| \geq |\Gamma_{\text{new}}(r, P)| - a$. In other words, more than $n - t - 1/\epsilon$ vertices in $V(P)^c$ satisfy

$$|\Gamma_{\text{new}}(r, P) \cap V(P)| > a,$$

so P is nice, contradicting our assumption. \square

Establishing a contradiction

In the following, let $P = (p_1, \dots, p_t)$ be a nice maximum rainbow path, that is, such that

$$|R(P)| > n - t - 1/\epsilon.$$

We will now try to reach a contradiction by constructing a rainbow path Q violating Lemma 1. We will start by proving the following useful proposition.

Proposition 8. Let A be any subset of $V(P)$. Then for any $k \in \mathbb{N}$, there are at most $1 + t/k$ vertices in A that do not have a k -successor in A (on P).

Proof. Let $X \subseteq A$ be the set of vertices in A that do not have a k -successor in A . Clearly, there can be only one vertex in A that does not have a t -successor in A . So for all but at most one vertex in X , there are k vertices following that vertex that are not in A . If we also count the vertices in X themselves, then in total we count at least $k(|X| - 1) + |X|$ vertices. Altogether, there are only t vertices in P . Therefore

$$k(|X| - 1) + |X| \leq t$$

and hence

$$|X| \leq \frac{t+k}{k+1} \leq \frac{t}{k} + 1.$$

This completes the proof. \square

Proposition 9. We have $|A(P) \cap B(P)| \leq \epsilon n$.

Proof. Suppose that

$$|A(P) \cap B(P)| > \epsilon n.$$

Let $k = \lceil t/(\epsilon n - 1) \rceil$. If $p_i \in A(P) \cap B(P)$, then by definition $p_{i-1} \in \Gamma_{\text{new}}(p_t, P)$. By Proposition 8, at most $1 + t/k$ vertices in $\Gamma_{\text{new}}(p_t, P)$ have no k -predecessor in $\Gamma_{\text{new}}(p_t, P)$ on P . Therefore, there are more than

$$\epsilon n - 1 - t/k \geq 0$$

Figure 2.4: Building a rainbow cycle for $p_i \in A(P) \cap B(P)$

vertices $p_i \in A(P) \cap B(P)$ such that p_{i-1} does have a k -predecessor p_j in $\Gamma_{\text{new}}(p_t, P)$. Take any such vertex; at least one of the edges $\{p_t, p_{i-1}\}$ and $\{p_t, p_j\}$ is colored differently from $\{p_1, p_{i+1}\}$. As shown in figure 2.4, there is a rainbow cycle C using this edge, of length at least

$$t - k - 1 \geq t - \lceil t/(\epsilon n - 1) \rceil - 1 \geq t - a/3.$$

Now we distinguish two cases.

- Case 1. There is a vertex $r \in R(P)$ such that for some $v \in V(C)$, we have $c(r, v) \notin c(C)$. Let $e \in E(C)$ be adjacent to v . Then it is easily verified that the path $C - e$ violates Lemma 1.
- Case 2. No vertex $r \in R(P)$ has a neighbor $v \in V(C)$ such that $c(r, v) \notin c(C)$. Let r be any vertex in $R(P)$. Since there are at most $|c(C)|$ vertices u such that $c(r, u) \in c(C)$, and because by assumption all of them are in $V(C)$, we have $c(r, p_i) \notin c(C)$. Then let $e \in E(C)$ be adjacent to p_{i+1} ; in this case the path $(C - e) \cup (p_{i+1}, p_i)$ violates Lemma 1.

Both cases result in a contradiction. \square

Proposition 10. There are at least ϵn vertices $p_i \in A(P)$ such that $c(p_i, p_{i+1}) \in c(p_1, R(P))$.

Proof. Let

$$X = \{p_i \in V(P) : c(p_i, p_{i+1}) \in c(p_1, R(P))\}.$$

First we show that $X \cap B(P) = \emptyset$. This is the case, because if $p_i \in X \cap B(P)$, then $c(p_i, p_{i+1}) = c(p_1, r)$ for some $r \in R(P)$, so $P' = (r, p_1, \dots, p_{i-1}, p_t, p_{t-1}, \dots, p_{i+1})$ is a rainbow path on t vertices. But since $r \in R(P)$, at least one vertex of $\Gamma_{\text{new}}(r, P')$ is not in $V(P')$, so P' cannot be maximum.

Hence $X \cap B(P) = \emptyset$. Then we can partition X as follows:

$$X = (X \cap (A(P) \cup B(P))^c) \cup (X \cap A(P)).$$

By Proposition 9, we have $|A(P) \cap B(P)| \leq \epsilon n$. So we get

$$\begin{aligned} |X \cap (A(P) \cup B(P))^c| &\leq |V(P)| - |A(P) \cup B(P)| \\ &= |V(P)| - |A(P)| - |B(P)| + |A(P) \cap B(P)| \\ &\leq t - n + t - n + t + \epsilon n \\ &= 3t - 2n + \epsilon n, \end{aligned}$$

and hence

$$|X \cap A(P)| = |X| - |X \cap (A(P) \cup B(P))^c| > 3n - 4t - 1/\epsilon - \epsilon n > \epsilon n,$$

using $|X| = |R(P)| > n - t - 1/\epsilon$. \square

By Proposition 8, at most $1 + \epsilon t$ vertices in $A(P)$ do not have a $\lceil 1/\epsilon \rceil$ -successor in $A(P)$. Thus at least $2\epsilon n - \epsilon t - 1 > 0$ vertices $p_i \in A(P)$ satisfy the following properties:

1. $c(p_i, p_{i+1}) \in c(p_1, R(P))$ and
2. p_i has a $\lceil 1/\epsilon \rceil$ -successor $p_j \in A(P)$.

Take any such vertices p_i and p_j .

Proposition 11. We have

$$|\Gamma_{\text{new}}(p_i, P) \cap \Gamma_{\text{new}}(p_j, P) \cap V(P)| > \epsilon n.$$

Proof. Because $p_i, p_j \in A(P)$, the paths $P_i = \rho_i \cdot P$ and $P_j = \rho_j \cdot P$ are maximum rainbow paths.

Both paths P_i and P_j end in the same vertex p_t . It will be useful to consider P_i and P_j as directed paths, so let E_i and E_j be the edge sets of P_i and P_j , but with the edges directed towards p_t . Because p_j is a $\lceil 1/\epsilon \rceil$ -successor of p_i , we have

$$|E_i \cup E_j| \leq t - 1 + \lceil 1/\epsilon \rceil \quad \text{and} \quad |E_i \cap E_j| \geq t - 1 - \lceil 1/\epsilon \rceil.$$

Since P_i and P_j are maximum rainbow paths, the colors in $c(p_t, V(P)^c)$ must appear on edges of P_i and P_j . Let Z be the set of edges in $E_i \cap E_j$ that are colored with colors in $c(p_t, V(P)^c)$. We have

$$|Z| \geq |c(p_t, V(P)^c)| - \lceil 1/\epsilon \rceil = n - t - \lceil 1/\epsilon \rceil.$$

If we write $P_i = (x_1, \dots, x_t)$ and $P_j = (y_1, \dots, y_t)$, then we can define

$$X = \{(x_i, x_{i+1}) : x_i \in A(P_i)\} \quad \text{and} \quad Y = \{(y_i, y_{i+1}) : y_i \in A(P_j)\}.$$

Note that it is sufficient to show that $|X \cap Y| > \epsilon n + 2$. By maximality of P_i and P_j ,

$$Z \cap X = Z \cap Y = \emptyset.$$

Therefore, we get

$$\begin{aligned} |Z \cup X \cup Y| &= |Z| + |X \cup Y| \\ &= |Z| + |X| + |Y| - |X \cap Y| \\ &\geq 3n - 3t - \lceil 1/\epsilon \rceil - |X \cap Y|. \end{aligned}$$

But $Z \cup X \cup Y \subseteq E_i \cup E_j$, so we have

$$|Z \cup X \cup Y| \leq |E_i \cup E_j| \leq t + \lceil 1/\epsilon \rceil,$$

and therefore

$$|X \cap Y| \geq 3n - 4t - 2\lceil 1/\epsilon \rceil > \epsilon n + 2,$$

which is what we needed to prove. \square

In the following, let us write

$$X = \Gamma_{\text{new}}(p_i, P) \cap \Gamma_{\text{new}}(p_j, P) \cap V(P).$$

We have just shown that $|X| \geq \epsilon n$. Because there are less than $1/\epsilon$ vertices of P between p_i and p_j , we can say that if n is large enough, then either there are at least $\epsilon n/3$ vertices of X preceding p_i on P (Case 1), or there are $\epsilon n/3$ vertices of X succeeding p_j (Case 2). Now we distinguish between the Cases 1 and 2.

Case 1. In this case, there are at least $\epsilon n/3$ vertices of X preceding p_i . Using Proposition 8, there are at most $1 + \epsilon t/10 < \epsilon n/10$ vertices in X that do not have a $\lceil 10/\epsilon \rceil$ -successor in X .

Therefore, at least

$$\frac{\epsilon n}{3} - \frac{\epsilon n}{10} = \frac{7\epsilon n}{30}$$

of the vertices of X preceding p_i have a $\lceil 10/\epsilon \rceil$ -successor in X . Note that there is a bijection between the vertices and their successors – to every vertex in X corresponds his closest successor in X . So by the same argument, of those successors, at least

$$\frac{7\epsilon n}{30} - \frac{\epsilon n}{10} = \frac{4\epsilon n}{30}$$

have themselves a $\lceil 10/\epsilon \rceil$ -successor in X .

Hence there is a vertex $u \in X$ preceding p_i on P that has two $\lceil 20/\epsilon \rceil$ -successors in X which also precede p_i . In particular, u has a $\lceil 20/\epsilon \rceil$ -successor v that precedes p_i and that satisfies $c(p_i, u) \neq c(p_j, v)$.

Then the path shown in figure 2.5 is a rainbow path starting in p_1 and ending in p_t , visiting at least $t - \lceil 1/\epsilon \rceil - \lceil 20/\epsilon \rceil$ vertices, that uses only two colors not in $c(P)$.

Case 2. As in the previous case, there is a vertex $v \in X$ succeeding p_j on P that has a $\lceil 20/\epsilon \rceil$ -successor w which satisfies $c(p_i, u) \neq c(p_j, v)$. The path shown in figure 2.6 is a rainbow path starting in p_1 and ending in p_t , visiting at least $t - \lceil 1/\epsilon \rceil - \lceil 20/\epsilon \rceil$ vertices, that uses only two colors not in $c(P)$.

Figure 2.5: Figure for Case 1

Figure 2.6: Figure for Case 2

Notice that by this construction, the path that we get is always such that some color in $c(p_1, R(P))$ is not used, namely, $c(p_i, p_{i+1})$. But since $t - \lceil 1/\epsilon \rceil - \lceil 20/\epsilon \rceil > t - a/3$, in both cases the resulting path violates Lemma 1. This is the desired contradiction.

2.4 Conclusion

We have proved that for every $\epsilon > 0$, every proper edge coloring of the graph K_n contains a rainbow path on

$$t > \left(\frac{3}{4} - \epsilon \right) n$$

vertices, assuming that n is larger than some value n_0 depending on ϵ . This is a significant improvement over Theorem 2.

In [?], the authors proved that every proper edge coloring of K_n contains Hamiltonian cycles on at least $(2/3 - o(1))n$ different colors. Clearly, the extension of a rainbow path of length t to a Hamiltonian cycle in K_n gives a Hamiltonian cycle using at least t colors. So as a bonus we get the following corollary to Theorem 3.

Corollary 1. In every proper edge coloring of K_n , there are Hamiltonian cycles using at least $(3/4 - o(1))n$ different colors.

3 Paths with Repeated Colors

3.1 Introduction

In this chapter, we take a look at a natural generalization of rainbow paths. Given a proper edge coloring of K_n , a rainbow path uses no color more than once; now we allow for *k-rainbow paths*, using every color at most a fixed number k of times. Note that 1-rainbow paths are just rainbow paths, so we have the following theorem.

Theorem 4 [?]. In every proper edge coloring of K_n , there is a 1-rainbow path on at least $(2n + 1)/3$ vertices.

Mostly, we are interested in an asymptotic statement: how does the length of a maximum k -rainbow path increase with k ? The following is a simple bound, although for simplicity we only prove it for powers of two.

Proposition 12. Let $n = 2^m$, for some $m \in \mathbb{N}$. In every proper edge coloring of K_n and for any $k \geq 1$, there is a k -rainbow path of length $(1 - 1/2^k)n - O(k)$.

Proof. We actually prove a stronger statement: that for every vertex v of K_n , there is a k -rainbow path starting in v of length $(1 - 1/2^k)n - O(k)$. The proof goes by induction on the number of vertices. In the base case of K_1 , there are no paths of nonzero length, so the claim is trivially satisfied.

Now assume that the claim is true for some value n and consider the graph K_{2n} . Let v and k be given. Starting in v , there is a rainbow path $P = (v, p_2, \dots, p_n)$ on n vertices in K_{2n} . This was noted after the proof of Proposition 2 in the previous chapter. Then we can invoke the induction hypothesis to get a $(k - 1)$ -rainbow path starting in p_n and avoiding the vertices of P , of length $(1 - 1/2^{k-1})n - O(k - 1)$. Appending the two paths together, we get a k -rainbow path of length

$$n - 1 + \left(1 - \frac{1}{2^{k-1}}\right)n - O(k - 1) = \left(1 - \frac{1}{2^k}\right)2n - O(k),$$

concluding the proof. □

This proof, while simple, already contains an important idea: that we can use $(k - 1)$ -rainbow paths to build k -rainbow paths. The rest of this chapter is about improving on this result, but first we define some notation.

3. PATHS WITH REPEATED COLORS This section contains lemmas on maximal k -rainbow paths

Consider the complete graph $K_n = (V, E)$ and a proper edge coloring c of K_n . If P is a k -rainbow path with respect to c , then let $C_0(P) = c(P)^c$ and for $i \in [k]$, let $C_i(P)$ be the set of colors used exactly i times on edges of P . Clearly, $c(E) = \bigcup_{i=0}^k C_i(P)$.

Furthermore, for a k -rainbow path $P = (p_1, \dots, p_t)$, we define

$$C_A(P) = \{c(p_i, p_{i+1}) : c(p_1, p_{i+1}) \notin C_k(P)\}.$$

Now we define what we mean by a maximal k -rainbow path. A k -rainbow path $P = (p_1, \dots, p_t)$ is *maximal* if it satisfies both

$$c(p_1, V(P)^c) \subseteq C_k(P). \quad (3.1)$$

and

$$c(p_t, V(P)^c) \subseteq C_k(P) \setminus C_A(P) \quad (3.2)$$

We shall see that every maximum length k -rainbow path is also maximal in this sense.

3.2 Two lemmas on maximal k -rainbow paths

Lemma 3. If P is a $(k-1)$ -rainbow path, then there is a maximal k -rainbow path P' with $|C_k(P')| \leq |V(P')| - |V(P)|$.

Proof. Let P be any $(k-1)$ -rainbow path. Then P is also a (non-maximal) k -rainbow path with $C_k(P) = \emptyset$. We will show that for any non-maximal k -rainbow path P , there is a k -rainbow path P' with $|V(P')| = |V(P)| + 1$ and $|C_k(P')| \leq |C_k(P)| + 1$. Since we cannot add vertices indefinitely, we will eventually get a maximal k -rainbow path with the required properties.

So if $P = (p_1, \dots, p_t)$ is a non-maximal k -rainbow path, then one of the following is the case.

Case 1. If $c(p_1, V(P)^c) \not\subseteq C_k(P)$, then there is an edge $\{p_1, r\} \in E(p_1, V(P)^c)$ colored with a color $c \notin C_k(P)$. Hence the path $P \cup \{p_1, r\}$ is a k -rainbow path and $C_k(P \cup \{p_1, r\}) = C_k(P) \cup \{c\}$.

Case 2. If $c(p_t, V(P)^c) \not\subseteq C_k(P)$, then we can proceed just as in the first case, after reversing the order of the vertices on P .

Case 3. If $c(p_t, V(P)^c) \subseteq C_k(P)$ but $c(p_t, V(P)^c) \not\subseteq C_k(P) \setminus C_A(P)$, then there are vertices $p_i \in V(P)$ and $r \in V(P)^c$ such that $c(p_i, p_{i+1}) = c(p_t, r) \in C_k(P)$ and, furthermore, $c(p_1, p_{i+1}) \notin C_k(P)$.

Recall that $\rho_i \cdot P$ is the path $(p_i, p_{i-1}, \dots, p_1, p_{i+1}, \dots, p_t)$. Then $\rho_i \cdot P$ is a k -rainbow path with $|C_k(\rho_i \cdot P)| \leq |C_k(P)|$ and $|V(\rho_i \cdot P)| = |V(P)|$. Moreover, $\rho_i \cdot P$ ends in p_t and we have $c(p_t, r) \in C_{k-1}(\rho_i \cdot P)$. This means that $c(p_t, V(\rho_i \cdot P)^c) \not\subseteq C_k(\rho_i \cdot P)$, and so can we proceed as in the second case.

3. PATHS WITH REPEATED COLORS The lemmas on maximal k -rainbow paths

This completes the proof of the lemma. Note that in addition, we have proved that every maximum length k -rainbow path is maximal. \square

Lemma 4. If P is a maximal k -rainbow path, then

$$|C_k(P)| \geq (k+1)n - (k+1)|V(P)|.$$

Proof. Let $P = (p_1, \dots, p_t)$ be a maximal k -rainbow path. If $C \subseteq C(E)$ is a set of colors, then we write

$$\mathcal{E}[C] = \{e \in E(P) : c(e) \in C\} = c^{-1}(C) \cap E(P)$$

for the set of edges of P colored with a color in C .

First, we would like to find a lower bound for $|C_A(P) \cap C_k(P)|$. Since every color appears at most k times on P , we have

$$k|C_A(P) \cap C_k(P)| \geq |\mathcal{E}[C_A(P) \cap C_k(P)]|.$$

By maximality condition (3.1), every vertex v with $c(p_1, v) \notin C_k(P)$ is in $V(P)$, so we have

$$|\mathcal{E}[C_A(P)]| \geq |C_A(P)| \geq n - 1 - |C_k(P)|.$$

Moreover,

$$|\mathcal{E}[C_k(P)^c]| = |E(P)| - |\mathcal{E}[C_k(P)]| = t - 1 - k|C_k(P)|.$$

Then we get

$$\begin{aligned} k|C_A(P) \cap C_k(P)| &\geq |\mathcal{E}[C_A(P) \cap C_k(P)]| \\ &= |\mathcal{E}[C_A(P)] \setminus \mathcal{E}[C_k(P)^c]| \\ &\geq |\mathcal{E}[C_A(P)]| - |\mathcal{E}[C_k(P)^c]| \\ &\geq n - 1 - |C_k(P)| - (t - 1 - k|C_k(P)|) \\ &= n - t + (k - 1)|C_k(P)|. \end{aligned}$$

By maximality condition (3.2),

$$\begin{aligned} k|c(p_t, V(P)^c)| &\leq k|C_k \setminus C_A(P)| \\ &= k|C_k(P)| - k|C_A(P) \cap C_k(P)| \\ &\leq k|C_k(P)| - n + t - (k - 1)|C_k(P)| \\ &= |C_k(P)| - n + t \end{aligned}$$

With $|c(p_t, V(P)^c)| = n - t$, we get

$$|C_k(P)| \geq (k+1)n - (k+1)t,$$

as claimed. \square

3.3 A theorem on the length of k -rainbow paths

Theorem 5. In every proper coloring of K_n and for any $k \geq 1$, there is a k -rainbow path on at least

$$\left(1 - \frac{2}{(k+2)!}\right)n$$

vertices.

Proof. The proof goes by induction on k .

The induction basis is provided for by Theorem 4, since

$$\frac{2n+1}{3} \geq \left(1 - \frac{2}{3!}\right)n.$$

In the induction step, assume that there is a $(k-1)$ -rainbow path using

$$t_{k-1} > \left(1 - \frac{2}{(k+1)!}\right)n$$

vertices. Then, by Lemma 1, there is a maximal k -rainbow path P with $|C_k(P)| \leq |V(P)| - t_{k-1}$. Using Lemma 2, we get

$$|C_k(P)| \geq (k+1)n - (k+1)|V(P)|,$$

so if we write t_k for $|V(P)|$, then

$$t_k - t_{k-1} \geq (k+1)n - (k+1)t_k,$$

or

$$(k+2)t_k \geq t_{k-1} + (k+1)n.$$

Using the induction hypothesis, we get

$$\begin{aligned} t_k &\geq \frac{t_{k-1} + (k+1)n}{k+2} \\ &> \frac{\left(1 - \frac{2}{(k+1)!}\right)n}{k+2} + \frac{(k+1)n}{k+2} \\ &= \frac{n}{k+2} + \frac{(k+1)n}{k+2} - \frac{2n}{(k+2)!} \\ &= \frac{(k+2)n}{k+2} - \frac{2n}{(k+2)!} \\ &= \left(1 - \frac{2}{(k+2)!}\right)n, \end{aligned}$$

so P is a k -rainbow path of sufficient length. □

3.4 Conclusion

From the statement of Theorem 5, it is easily seen that for fixed n , the number of vertices not included in a maximum k -rainbow path is in the order of $1/k!$. This is clearly an improvement over Proposition 12, which only shows that the number of vertices not included in a maximum k -rainbow path is in the order of $1/2^k$. The growth provided by Theorem 5 is asymptotically faster.

In the proof, we essentially used a generalized version of the argument made by Gyárfás and Mhalla in [?], a discussion of which can be found in the previous chapter. In this light, it might be interesting to generalize the techniques used in the proof of the bound of $(3/4 - o(1))n$ for the length of maximum rainbow paths in K_n , and to apply them to k -rainbow paths.

4 Conclusion

In the preceding chapters, we have derived two novel results on the existence of certain paths in proper edge colorings of the complete graph K_n .

Most importantly, we have shown that in every proper edge coloring of K_n there is a rainbow path of length at least

$$\left(\frac{3}{4} - o(1)\right)n.$$

This result improves on the previously best known bound of $2n/3$ proved by Gyárfás and Mhalla in [?]. As a corollary, there are Hamiltonian cycles using at least $(3/4 - o(1))n$ colors; here we improve on a result by Akbari, Etesami, Mahini and Mahmoody [?].

Moreover, we have proved that in every proper edge coloring of K_n , there is a k -rainbow path on at least

$$\left(1 - \frac{2}{(k+2)!}\right)n$$

vertices, for any $k > 0$. Thus, for fixed n , the number of vertices not included in a maximum k -rainbow path decreases with k faster than any exponential function, and hence asymptotically faster than what we got using a naive approach.

We believe that the techniques used in the proof of the first result, which relied heavily on pigeonhole-style arguments, do not immediately lend themselves to proving stronger bounds. Moreover, the proof itself does not seem to reveal deep insights into the structure of rainbow paths, leading us to believe that different methods will have to be used to prove the existence of rainbow paths of length $n - o(n)$.

However, it seems likely that applying the same methods, suitably generalized, to the problem of k -rainbow paths in K_n might prove to be fruitful. Indeed, we would expect the resulting bound to be asymptotically stronger (in k , for fixed n) than our bound, whose proof relied on comparatively simple techniques.

